

Ashtekar's variables without spin

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Abstract

Ashtekar's variables are shown to arise naturally from a $3 + 1$ split of general relativity in the Einstein-Cartan formulation. Thereby spinors are exorcised.

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Ashtekar's variables [1] have already received much attention [2] and need no introduction. It is intended to show that they arise naturally from the Einstein-Cartan formulation of general relativity without the use of spinors. We follow the notations of reference [3] where a detailed presentation of the Einstein-Cartan theory may be found. Cartan's key idea is to describe a metric by an orthonormal frame of vector fields e_a , $a = 0, 1, 2, 3$:

$$g(e_a, e_b) = \eta_{ab} \quad (1)$$

$$(\eta_{ab}) = \text{diag}(+1, -1, -1, -1) \quad (2)$$

or its dual frame of 1-forms e^a , $a = 0, 1, 2, 3$:

$$e^a(e_b) = \delta^a_b. \quad (3)$$

Of course two frames e_a and e'_a related by a Lorentz transformation $\Lambda \in SO(1, 3)$

$$e'_a = \Lambda^{-1b}_a e_b, \quad (4)$$

$$\Lambda^T \eta \Lambda = \eta \quad (5)$$

describe the same metric. In order to construct an invariant action one introduces a connection ω , i.e. a 1-form on space-time M with values in the Lie algebra $so(1, 3)$, which then preserves the metric under parallel transport. Under a change of frame (4) the connection transforms as

$$\omega' = \Lambda \omega \Lambda^{-1} + \Lambda d\Lambda^{-1}. \quad (6)$$

In these variables e and ω the Einstein-Hilbert action reads:

$$S_{EH}(e, \omega) := \frac{-1}{32\pi G} \int_M R^a_{\ b} \eta^{b' b} \wedge e^c \wedge e^d \epsilon_{abcd} \quad (7)$$

where R is the curvature of ω , a 2-form with values in $so(1, 3)$:

$$R = d\omega + \frac{1}{2} [\omega, \omega] \quad (8)$$

and ϵ_{abcd} is completely antisymmetric with $\epsilon_{0123} = 1$. Variations with respect to the (dual) frame give the Einstein equations

$$R^{ab} \wedge e^d \epsilon_{abcd} = 0 \quad (9)$$

while varying the connection and partial integration yields vanishing torsion

$$T^a := De^a := de^a + \omega^a_b \wedge e^b = 0. \quad (10)$$

These linear equations in ω can be solved uniquely to express the connection as a function of the frame components and their first derivatives, the so-called Riemannian connection. Choose a time coordinate $t : M \rightarrow \mathbb{R}$

$${}^{\star}g(dt, dt) > 0 \quad (11)$$

${}^{\star}g$ being the induced metric for 1-forms. A 3+1 split is a parametrization of all metrics on M by starting from a parametrization of all metrics on the 3-surfaces \sum_t of “simultaneity”. In the frame formulation this imposes the use of vector fields rather than 1-forms. Let $e_{\bar{a}}$, $\bar{a} = 1, 2, 3$ be three linearly independent vector fields tangent to \sum_t :

$$dt(e_{\bar{a}}) = 0, \quad \bar{a} = 1, 2, 3. \quad (12)$$

Modulo $SO(3)$ rotations they parametrize the metrics on \sum_t . This parametrization is completed to include all metrics on M by a vector field e_0 pointing towards the future:

$$dt(e_0) > 0. \quad (13)$$

In coordinates $x^0 = t$, $x^{\bar{\mu}}$, $\bar{\mu} = 1, 2, 3$ we have

$$e_a = (\gamma^{-1})^{\mu}_a \frac{\partial}{\partial x^{\mu}}, \quad (14)$$

$$(\gamma^{-1})^0_{\bar{a}} = 0 \quad \bar{a} = 1, 2, 3. \quad (15)$$

Consequently there are $1+3+9=13$ variables $(\gamma^{-1})^0_0$, $(\gamma^{-1})^{\bar{\mu}}_0$, $(\gamma^{-1})^{\bar{\mu}}_{\bar{a}}$. From the 9 $(\gamma^{-1})^{\bar{\mu}}_{\bar{a}}$ we still have to subtract 3 degrees of freedom for the $SO(3)$ transformations and finally we remain with 10 variables parametrizing all metrics. This parametrization was introduced in 1963 by Schwinger [4] under the name “time gauge”. For later convenience we rename the components of the frame

$$dx^{\mu}(e_a) = (\gamma^{-1})^{\mu}_a =: 8\pi G \det(\pi^{\bar{\nu}}_{\bar{b}}) \begin{pmatrix} \frac{1}{N} & 0 \\ \frac{n^{\bar{\mu}}}{N} & \pi^{\bar{\mu}}_{\bar{a}} \end{pmatrix}. \quad (16)$$

Let

$$\omega^a_b =: \omega^a_{b\mu} dx^{\mu} \quad (17)$$

be the components of the connection. From now on we raise Latin (Lorentz-)indices with $\eta^{ab} = \eta_{ab}$. A bar over a Latin or Greek (coordinate) index indicates that it only takes spatial values. We can then write the Einstein-Hilbert action as:

$$\begin{aligned} S_{EH}(e, \omega) = & \int_{-\infty}^{+\infty} dt \int_{\sum_t} dx^1 dx^2 dx^3 \left[\left(n^{\bar{\nu}} \pi^{\bar{\mu}}_{\bar{b}} - n^{\bar{\mu}} \pi^{\bar{\nu}}_{\bar{b}} \right) \left(\partial_{\bar{\mu}} \omega^{0\bar{b}}_{\bar{\nu}} - \omega^{0\bar{c}}_{\bar{\mu}} \omega^{\bar{c}\bar{b}}_{\bar{\nu}} \right) \right. \\ & - \frac{1}{2} N \left(\pi^{\bar{\mu}}_{\bar{a}} \pi^{\bar{\nu}}_{\bar{b}} - \pi^{\bar{\nu}}_{\bar{a}} \pi^{\bar{\mu}}_{\bar{b}} \right) \left(\partial_{\bar{\mu}} \omega^{\bar{a}\bar{b}}_{\bar{\nu}} - \omega^{\bar{a}\bar{c}}_{\bar{\mu}} \omega^{\bar{c}\bar{b}}_{\bar{\nu}} - \omega^{0\bar{a}}_{\bar{\mu}} \omega^{0\bar{b}}_{\bar{\nu}} \right) \\ & \left. - \pi^{\bar{\nu}}_{\bar{b}} \left(\partial_0 \omega^{0\bar{b}}_{\bar{\nu}} - \partial_{\bar{\nu}} \omega^{0\bar{b}}_0 - \omega^{0\bar{c}}_0 \omega^{\bar{c}\bar{b}}_{\bar{\nu}} + \omega^{0\bar{c}}_{\bar{\nu}} \omega^{\bar{c}\bar{b}}_0 \right) \right]. \quad (18) \end{aligned}$$

In the variables N , $n^{\bar{\mu}}$, $\pi^{\bar{\mu}}_a$ and ω^{ab}_{μ} the action is polynomial. Furthermore the requirement that the three vector fields $e_{\bar{a}}$ be linearly independent may be dropped allowing

also degenerate $\pi_{\bar{a}}^{\bar{\mu}}$. The only dynamical variables are the nine functions $\omega_{\bar{\nu}}^{0\bar{b}}$ which are related to the extrinsic curvature of Σ_t . Their time derivative only appears linearly in the action and their conjugate momenta are the nine $-\pi_{\bar{a}}^{\bar{\mu}}$. Variation with respect to N , $n^{\bar{\mu}}$ and $\pi_{\bar{a}}^{\bar{\mu}}$ yields the Einstein equations, while varying ω and a partial integration gives again the torsion zero equation in the following form:

$$\delta\omega_{\bar{0}}^{0\bar{b}} : \quad D_{\bar{\nu}}\pi_{\bar{b}}^{\bar{\nu}} := \partial_{\bar{\nu}}\pi_{\bar{b}}^{\bar{\nu}} + \omega_{\bar{b}}^{\bar{c}}{}_{\bar{\nu}}\pi_{\bar{c}}^{\bar{\nu}} = 0. \quad (19)$$

D_{μ} is the $SO(3)$ -covariant derivative with respect to $\omega_{\bar{b}\mu}^{\bar{a}}$.

$$\delta\omega_{\bar{0}}^{\bar{a}\bar{b}} : \quad \omega_{\bar{a}\bar{\nu}}^0\pi_{\bar{b}}^{\bar{\nu}} - \omega_{\bar{b}\bar{\nu}}^0\pi_{\bar{a}}^{\bar{\nu}} = 0, \quad (20)$$

$$\begin{aligned} \delta\omega_{\bar{\mu}}^{\bar{a}\bar{b}} : \quad & (\omega_{\bar{a}\bar{0}}^0\pi_{\bar{b}}^{\bar{\mu}} - \omega_{\bar{b}\bar{0}}^0\pi_{\bar{a}}^{\bar{\mu}}) + \omega_{\bar{a}\bar{\nu}}^0(n^{\bar{\nu}}\pi_{\bar{b}}^{\bar{\mu}} - n^{\bar{\mu}}\pi_{\bar{b}}^{\bar{\nu}}) - \omega_{\bar{b}\bar{\nu}}^0(n^{\bar{\nu}}\pi_{\bar{a}}^{\bar{\mu}} - n^{\bar{\mu}}\pi_{\bar{a}}^{\bar{\nu}}) \\ & - D_{\bar{\nu}}(N\pi_{\bar{a}}^{\bar{\nu}}\pi_{\bar{b}}^{\bar{\mu}} - N\pi_{\bar{b}}^{\bar{\nu}}\pi_{\bar{a}}^{\bar{\mu}}) = 0, \end{aligned} \quad (21)$$

$$\delta\omega_{\bar{\mu}}^{0\bar{b}} : \quad D_0\pi_{\bar{b}}^{\bar{\mu}} + D_{\bar{\nu}}(n^{\bar{\nu}}\pi_{\bar{b}}^{\bar{\mu}} - n^{\bar{\mu}}\pi_{\bar{b}}^{\bar{\nu}}) + \omega_{\bar{\nu}}^{0\bar{c}}(N\pi_{\bar{c}}^{\bar{\nu}}\pi_{\bar{b}}^{\bar{\mu}} - N\pi_{\bar{c}}^{\bar{\mu}}\pi_{\bar{b}}^{\bar{\nu}}) = 0. \quad (22)$$

Now, if we replace the 6 real-valued 1-forms ω^{ab} by the 3 complex-valued 1-forms $\chi^{\bar{a}\bar{b}}$

$$\chi^{\bar{a}\bar{b}} := \omega^{\bar{a}\bar{b}} + i\epsilon_{0\bar{c}}^{\bar{a}\bar{b}}\omega^{0\bar{c}} \quad (23)$$

and if we denote by \mathcal{D}_{μ} the covariant derivative with respect to $\chi^{\bar{a}\bar{b}}_{\mu}$, then the 6 real equations (19) and (20) combine into 3 complex equations:

$$\mathcal{D}_{\bar{\nu}}\pi_{\bar{b}}^{\bar{\nu}} = 0. \quad (24)$$

Likewise, (21) and (22) condense to

$$\mathcal{D}_0\pi_{\bar{b}}^{\bar{\mu}} + \mathcal{D}_{\bar{\nu}}(n^{\bar{\nu}}\pi_{\bar{b}}^{\bar{\mu}} - n^{\bar{\mu}}\pi_{\bar{b}}^{\bar{\nu}}) + i\epsilon_{0\bar{b}}^{\bar{c}\bar{d}}\mathcal{D}_{\bar{\nu}}(N\pi_{\bar{c}}^{\bar{\nu}}\pi_{\bar{d}}^{\bar{\mu}}) = 0. \quad (25)$$

Samuel [5] and Jacobson and Smolin [6] have remarked that these complex equations can conveniently be obtained by adding to the Einstein-Hilbert action a piece that vanishes “on shell”: Let

$$\phi := \frac{1}{2}(\omega + i\tilde{\omega}) \quad (26)$$

where $\tilde{\omega}$ is the dual connection:

$$\tilde{\omega}^{ab} := \frac{1}{2}\epsilon^{ab}{}_{cd}\omega^{cd}. \quad (27)$$

Note the difference between dual and Hodge star. ϕ is complex but self-dual in the sense that

$$\tilde{\phi} = -i\phi. \quad (28)$$

Consequently

$$\phi^{\bar{a}\bar{b}} = i\epsilon^{\bar{a}\bar{b}}_{0\bar{c}}\phi^{0\bar{c}}. \quad (29)$$

Replacing the real ω by the complex ϕ in the Einstein-Hilbert action amounts to adding to the real Einstein-Hilbert action a purely imaginary piece:

$$S_{EH}(e, \phi) = \frac{1}{2}S_{EH}(e, \omega) + \frac{i}{2}S_I(e, \omega) \quad (30)$$

where

$$S_I(e, \omega) := \frac{-1}{32\pi G} \int_M \tilde{R}^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} = \frac{-1}{16\pi G} \int_M R_{cd} \wedge e^c \wedge e^d. \quad (31)$$

Indeed the curvature of ϕ is

$$d\phi + \frac{1}{2}[\phi, \phi] = \frac{1}{2}R + \frac{i}{2}\tilde{R} \quad (32)$$

where \tilde{R} is the dual of the curvature R of ω . The field equations of the complex action $S_{EH} + iS_I$ are the same as the field equations of the real Einstein-Hilbert action alone, because varying S_I with respect to the frame yields the Bianchi identity:

$$0 = DT = R \wedge e \quad (33)$$

and variation with respect to the connection gives again vanishing torsion. We define the connection χ , a 1-form on M with values in $so(3, \mathbb{C})$ by

$$\chi^{\bar{a}\bar{b}} := 2\phi^{\bar{a}\bar{b}} = \phi^{\bar{a}\bar{b}} + i\epsilon^{\bar{a}\bar{b}}_{0\bar{c}}\phi^{0\bar{c}} = \omega^{\bar{a}\bar{b}} + i\epsilon^{\bar{a}\bar{b}}_{0\bar{c}}\omega^{0\bar{c}} \quad (34)$$

and denote by \mathcal{D} its covariant derivative and by F its curvature

$$F^{\bar{a}}_{\bar{b}} := d\chi^{\bar{a}}_{\bar{b}} + \chi^{\bar{a}}_{\bar{c}}\chi^{\bar{c}}_{\bar{b}} =: \frac{1}{2}F^{\bar{a}}_{\bar{b}\mu\nu}dx^\mu \wedge dx^\nu. \quad (35)$$

Then, up to a surface term from the last expression on the right-hand side of equation (36), the action can be written:

$$\begin{aligned} S_{EH}(e, \omega) + iS_I(e, \omega) = & \int_{-\infty}^{+\infty} dt \int_{\Sigma_t} d^3x \left[\frac{i}{2} n^{\bar{\nu}} \pi^{\bar{\mu}}_{\bar{a}} \epsilon^{\bar{0}\bar{a}}_{\bar{c}\bar{d}} F^{\bar{c}\bar{d}}_{\bar{\mu}\bar{\nu}} - \frac{1}{2} N \pi^{\bar{\mu}}_{\bar{a}} \pi^{\bar{\nu}}_{\bar{b}} F^{\bar{a}\bar{b}}_{\bar{\mu}\bar{\nu}} \right. \\ & \left. - \frac{i}{2} \epsilon^{\bar{0}\bar{a}}_{\bar{c}\bar{d}} \partial_0 \chi^{\bar{c}\bar{d}}_{\bar{\nu}} \pi^{\bar{\nu}}_{\bar{a}} - \frac{i}{2} \epsilon^{\bar{0}\bar{a}}_{\bar{c}\bar{d}} \chi^{\bar{c}\bar{d}}_0 \mathcal{D}_{\bar{\nu}} \pi^{\bar{\nu}}_{\bar{a}} \right] \end{aligned} \quad (36)$$

Variations with respect to $\chi^{\bar{a}\bar{b}}_0$ and $\chi^{\bar{a}\bar{b}}_{\bar{\mu}}$ yield again equations (24) and (25), respectively. Note that now due to the additional term iS_I , all 18 components of $\chi^{\bar{a}\bar{b}}_{\bar{\mu}}$ are dynamical and their momenta are

$$p^{\bar{\mu}}_{\bar{a}\bar{b}} := -i\epsilon^{\bar{0}\bar{c}}_{\bar{a}\bar{b}}\pi^{\bar{\mu}}_{\bar{c}} \quad (37)$$

considered as complex variables. On the other hand, $n^{\bar{\mu}}$, N and $\chi^{\bar{c}\bar{d}}_0$ remain Lagrange multipliers. Since the time derivatives of $\chi^{\bar{c}\bar{d}}_{\bar{\nu}}$ appear only linearly in the action, the Hamiltonian is obtained by deleting the third term in equation (36) and changing all signs:

$$H = \int_{\Sigma_t} d^3x \left[\frac{1}{2} n^{\bar{\nu}} p^{\bar{\mu}}_{\bar{c}\bar{d}} F^{\bar{c}\bar{d}}_{\bar{\mu}\bar{\nu}} - \frac{1}{8} N \epsilon^{\bar{c}\bar{d}}_{0\bar{a}} p^{\bar{\mu}}_{\bar{c}\bar{d}} \epsilon^{\bar{r}\bar{s}}_{0\bar{b}} p^{\bar{\nu}}_{\bar{r}\bar{s}} F^{\bar{a}\bar{b}}_{\bar{\mu}\bar{\nu}} - \frac{1}{2} \chi^{\bar{c}\bar{d}}_0 \mathcal{D}_{\bar{\nu}} p^{\bar{\nu}}_{\bar{c}\bar{d}} \right]. \quad (38)$$

Although the Hamiltonian is complex, its time evolution carries real initial data π, ω into real data. In conclusion, we have recovered Ashtekar's variables which in the spinless formulation are given by $\chi^{\bar{a}\bar{b}}_{\bar{\mu}}$ and $p^{\bar{\mu}}_{\bar{a}\bar{b}}$.

References

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